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Integrability of the vacuum Einstein equation

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The Inverse Scattering Spectral Transform

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The **Inverse Scattering Transform** (IST) is one of the most important developments in mathematical physics in the past 40 years. The method is a non-linear analogue, and in some sense generalization, of the Fourier transform, ...

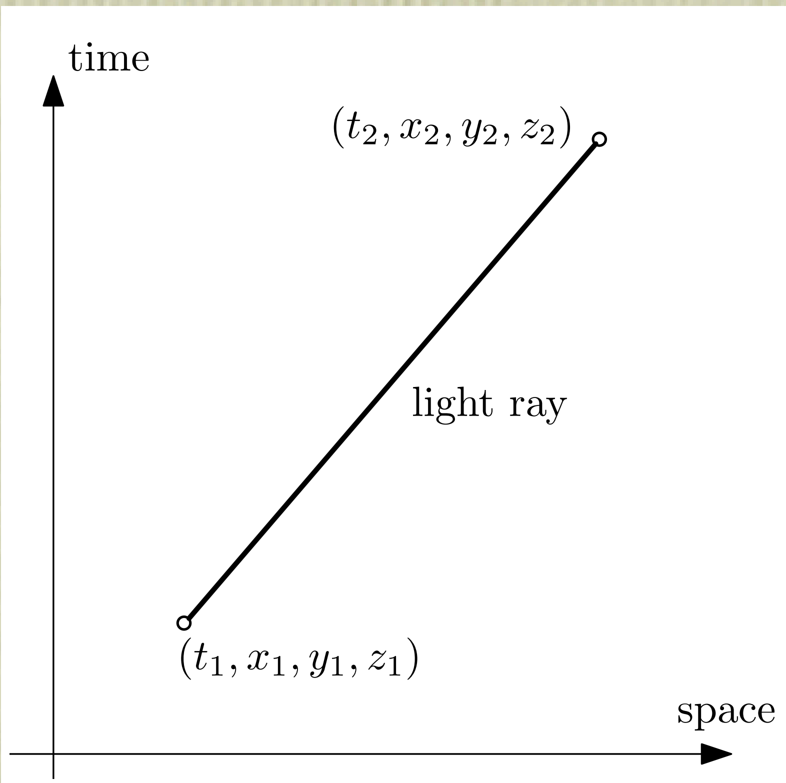
The Fourier Transform

$$
\begin{tikzcd}
\text{linear PDE} \arrow[r, "\mathcal{F}"] \arrow[d, dashed] & \text{ODE} \arrow[d] \\
\text{solution} & \mathcal{F}[\text{solution}] \arrow[l, "\mathcal{F}^{-1}"']
\end{tikzcd}
$$

The Inverse Scattering Transform

$$
\begin{tikzcd}
\begin{array}{c}\text{nonlinear PDE}\\ F[u]=0\end{array} \arrow[r] \arrow[d, dashed] & \begin{array}{c}\text{2 linear PDEs}\\ \begin{array}{c}\textit{Lax}\\ \textit{pair}\end{array}\left\{\begin{array}{l} L_1(\psi;u;t,x,\lambda)=0\\ L_2(\psi;u;t,x,\lambda)=0\end{array}\right.\end{array} \arrow[d] \\
\begin{array}{c}\text{solution}\\ u=u(t,x)\end{array} & \begin{array}{c}\text{solution}\\ \psi(t,x,\lambda)\end{array} \arrow[l]
\end{tikzcd}
$$

The theory of relativity



$$\sqrt{(dx)^2 + (dy)^2 + (dz)^2} = dt \iff ds^2 = 0$$

Denote $x^0 = t, x^1 = x, x^2 = y, x^3 = z$

and $\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The spacetime interval $ds^2 := g_{\mu\nu} dx^\mu dx^\nu$

<u>The equivalence principle:</u> gravity affects all bodies in the same way, independently of their composition.

In empty space, Spacetime curves according to Einstein's vacuum equation

Ricci Curvature $R_{\mu\nu} = 0$

The Belinski-Zakharov transform

$$ds^2 = f(-dt^2 + dr^2) + g_{ab}dx^a dx^b \qquad a,b = 1,2 \quad x^a = (\phi, z)$$

In matrix form, $g_{\mu\nu} = \begin{bmatrix} -f & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$

light-cone coordinates

$$t = \zeta - \eta \qquad r = \zeta + \eta$$

$$R_{\mu\nu} = 0$$

$$\underbrace{\left(\alpha g_{,\zeta} g^{-1}\right)}_{-A}{}_{,\eta} + \underbrace{\left(\alpha g_{,\eta} g^{-1}\right)}_{B}{}_{,\zeta} = 0 \qquad \det(g) = \alpha^2$$

f is easily integrated once g is known

$$\left.\begin{aligned} D_1 &= \partial_\zeta \boxed{- \frac{2\alpha_{,\zeta}\lambda}{\lambda - \alpha}\partial_\lambda} \\ D_2 &= \partial_\eta \boxed{+ \frac{2\alpha_{,\eta}\lambda}{\lambda + \alpha}\partial_\lambda} \end{aligned}\right\} \textit{Commuting operators} \qquad \left.\begin{aligned} D_1\psi &= \frac{A}{\lambda - \alpha}\psi \\ D_2\psi &= \frac{B}{\lambda + \alpha}\psi \end{aligned}\right\} \textit{Lax pair}$$

In the limit $\lambda \to 0$ get $g(\zeta, \eta) = \psi(0, \zeta, \eta)$

The dressing method

1

Take a particular "background" solution $g^{(0)}(\zeta,\eta)$

2

$$\underbrace{\begin{aligned} D_1\psi_0 &= \frac{A_0}{\lambda-\alpha}\psi_0 \\ D_2\psi_0 &= \frac{B_0}{\lambda+\alpha}\psi_0 \end{aligned}}^{\text{can skip this step}} \Longrightarrow \text{get } \psi_0$$

3

Search for a solution of the form $\psi = \chi\psi_0$

χ : the dressing matrix

$$\chi = I + \sum_{s=1}^{N} \frac{R_s}{\lambda - \mu_s}$$

4

Get the N-soliton solution

$$g^{(N)} = \chi(0)g^{(0)}$$

'Cylindrical' metrics: $ds^2 = f^{(0)}(-dt^2 + dr^2) + e^{\Lambda^{(0)}}(rd\phi)^2 + e^{-\Lambda^{(0)}}dz^2$

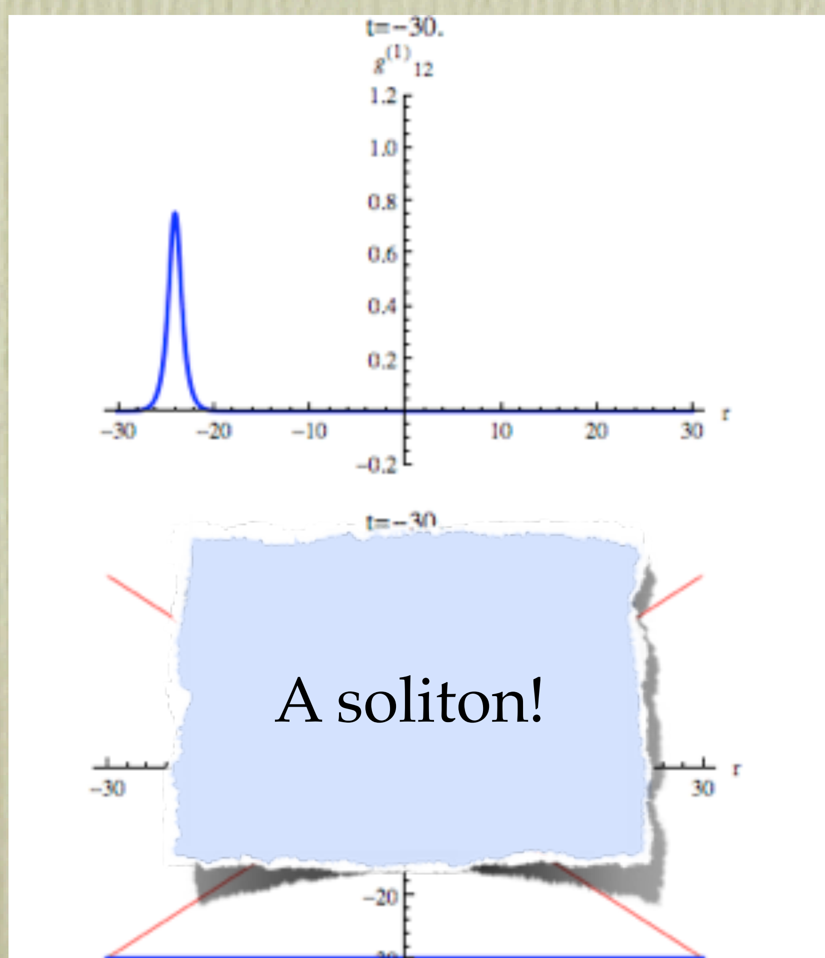
Solitons (one soliton)

Symmetric chiral equation

$$(g_{,\zeta}g^{-1})_{,\eta} + (g_{,\eta}g^{-1})_{,\zeta} = 0$$

$$g^{(0)} = \begin{bmatrix} e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}$$

$$\mu_s = \text{constant}$$

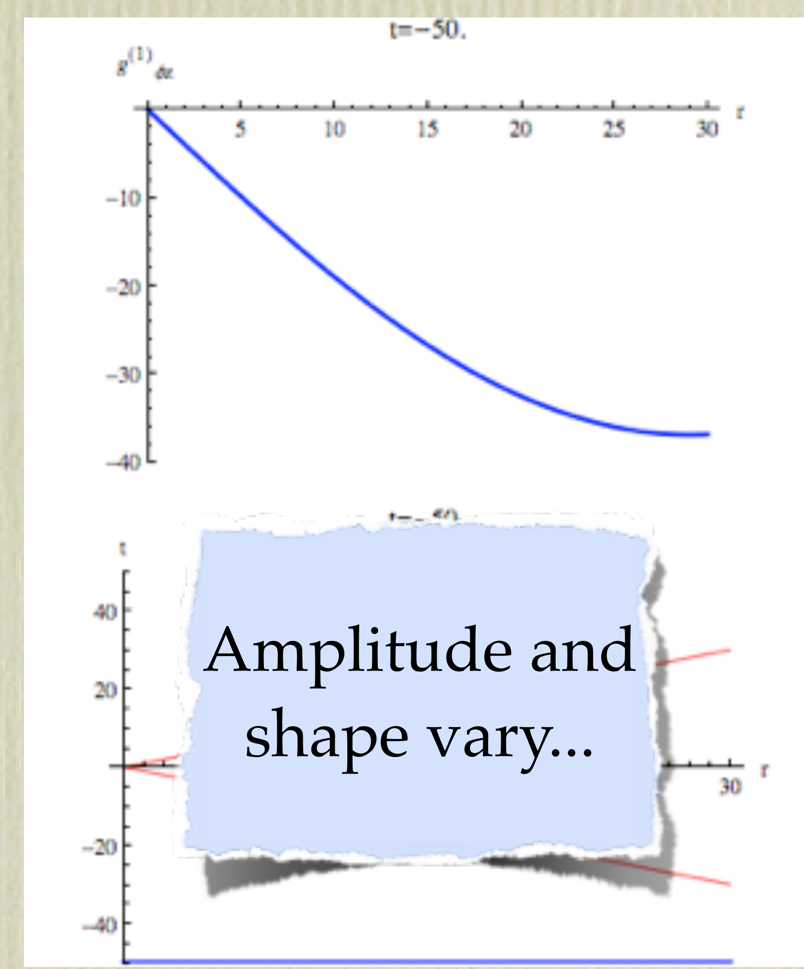


Einstein equations

$$\left(\alpha g_{,\zeta}g^{-1}\right)_{,\eta} + \left(\alpha g_{,\eta}g^{-1}\right)_{,\zeta} = 0$$

$$g^{(0)} = \begin{bmatrix} r^2 e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}$$

$$\mu_s = \omega_s - t \pm \sqrt{(\omega_s - t)^2 - r^2}$$



Solitons (two solitons)

For physical reasons, one would like the metric to be *asymptotically flat*.

The one-soliton solution is never asymptotically flat.

Pole $\mu_s = \omega_s - t \pm \sqrt{(\omega_s - t)^2 - r^2}$

$+ =$ soliton
$- =$ anti-soliton

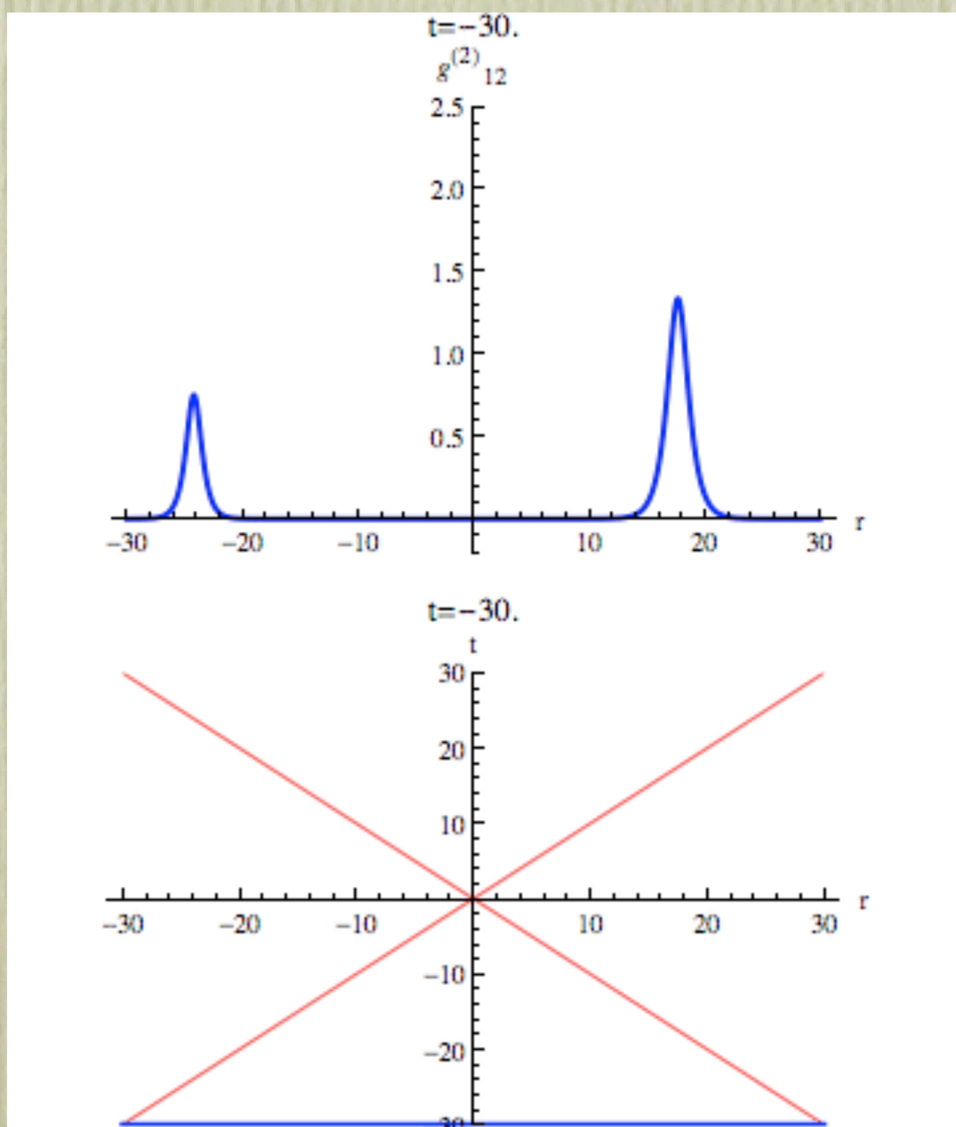
Theorem: If $\Lambda^{(0)} \xrightarrow{|t| \to \infty} 0$ *and (# of solitons) = (# of anti-solitons) then the solution is asymptotically flat as* $|t| \to \infty$.

Both S-S or AS-AS are never asymptotically flat.

Solitons (two solitons)

Symmetric chiral equation

$$g^{(0)} = \begin{bmatrix} e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}$$



Einstein equations

$$g^{(0)} = \begin{bmatrix} r^2 e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}$$

$$\Lambda^{(0)} = 0 \qquad \text{(flat)}$$

Thank you!

Supplemental Slides